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Matthias Kouakou, Alexis Tchoudjem

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M.K. Kouakou and A. Tchoudjem
Université de Cocody
UFR-Mathématiques et Informatique
22 BP 582 Abidjan 22 Côte d'Ivoire
e-mail : makonankouakou@yahoo.fr
and
Université Claude Bernard Lyon I
Institut Camille-Jordan
69622 Villeurbanne cedex France
tchoudjem@math.univ-lyon1.fr

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Abstract

Let $A_1 := \mathbb{k}[t, \partial]$ be the first algebra over a field $\mathbb{k}$ of characteristic zero. One can associate to each right ideal I of A_1 its Stafford subgroup, which is a subgroup of $\mathrm{Aut}_{\mathbb{k}}(A_1)$, the automorphism group of the ring A_1. In this article we show that each Stafford subgroup is equal to its normalizer. For that, we study when the Stafford subgroup of a right ideal of A_1 contains a given Stafford subgroup.

Introduction

Let $\mathbb{k}$ be a commutative field of characteristic zero. We note A_1 the first Weyl agebra over $\mathbb{k}$ *i.e.* :

$$A_1 := A_1(\mathbb{k}) = \mathbb{k}[t, \partial]$$

where ∂, t are related by $\partial t - t\partial = 1$.

Definition 1 *For a right ideal I of A_1, the Stafford subgroup associated to I is :*

$$H(I) := \{\sigma \in Aut_{\Bbbk}(A_1) : \sigma(I) \simeq I\}$$

(where the symbol "$\simeq$" means "$\sigma(I)$ is isomorphic to I as a right-A_1-module").

By [5], it is known that each subgroup $H(I)$ is isomorphic to an automorphism group $\mathrm{Aut}_{\Bbbk}(\mathcal{D}(X))$, where $\mathcal{D}(X)$ is the $\Bbbk-$algebra of differential operators over an algebraic affine curve X.

A natural question is :

"are the Stafford subgroups normal in $\mathrm{Aut}_{\Bbbk}(A_1)$" ?

The answer is no.

Stafford showed that if X_2 is the famous algebraic plane curve defined by the equation :

$$x^2 = y^3$$

and if I_2 is the right ideal of A_1 :

$$I_2 := \Big\{ d \in A_1 \ : \ d(\Bbbk[t]) \subseteq \Bbbk[t^2, t^3] \Big\}$$

then the subgroup $H(I_2)$ is isomorphic to $\mathrm{Aut}_{\Bbbk}(\mathcal{D}(X_2))$ and is equal to its own normalizer in $\mathrm{Aut}_{\Bbbk}(A_1)$.

We will show in this paper that the subgroup $H(I)$ is equal to its own normalizer for all right ideal I of A_1.

We begin by giving some definitions and by fixing some notations that will be used in this paper.

1 Definitions and some properties

The ring A_1 contains the subrings $R := \Bbbk[t]$ and $S := \Bbbk[\partial]$. It is well known that A_1 is a two-sided noetherian integral domain. Since the characteristic of $\Bbbk$ is zero, A_1 is also hereditary (*cf* [4]) *i.e.* every non zero right ideal of A_1 is a projective right-A_1-module.

The ring A_1 has a quotient divison ring, denoted by Q_1. For any finitely generated right-submodule M of Q_1, the dual M^*, as a left-A_1-module will be identified with the set $\{u \in Q_1 : uM \subseteq A_1\}$, and $End_{A_1}(M)$ with the set $\{d \in Q_1 : dM \subseteq M\}$ (*cf* [5]).

The division ring Q_1 contains the subrings $D := \mathbb{k}(t)[\partial]$ and $E := \mathbb{k}(\partial)[t]$. The elements of D are $\mathbb{k}$-linear endomorphisms of $\mathbb{k}(t)$. More precisely, if $d := a_n\partial^n + ... + a_1\partial + a_0$ for some $a_i \in \mathbb{k}(t)$ and if $h \in \mathbb{k}(t)$, then :

$$d(h) := a_n h^{(n)} + ... + a_1 h^{(1)} + a_0 h \ ,$$

where $h^{(i)}$ denotes the i-th derivative of h and $a_i h^{(i)}$ is a product in $\mathbb{k}(t)$. We note that:

$$\forall\ d,\ d' \in \mathbb{k}(t)[\partial],\ \forall\ h \in \mathbb{k}(t),\ (dd')(h) = d(d'(h))\ .$$

For V and W two vector subspaces of $\mathbb{k}(t)$, we set :

$$\mathcal{D}(V, W) := \{d \in \mathbb{k}(t)[\partial] : d(V) \subseteq W\}\ .$$

Notice that $\mathcal{D}(R, V)$ is a right A_1-submodule of Q_1 and $\mathcal{D}(V, R)$ is a left A_1-submodule of Q_1. If moreover $V \subseteq R$, then $\mathcal{D}(R, V)$ is a right ideal of A_1. When $V = R$, one has $\mathcal{D}(R, R) = A_1$.

If I is a right ideal of A_1, we set :

$$I \star 1 := \{d(1)\ :\ d \in I\}\ .$$

It is clear $I \star 1$ is a $\mathbb{k}-$vector subspace of $\mathbb{k}[t]$ and that :

$$I \subseteq \mathcal{D}(R, I \star 1)\ .$$

The inclusions $A_1 \subset k(\partial)[t]$ and $A_1 \subset k(t)[\partial]$ show that, at least, two notions of degree can be defined on A_1 : the degree in "t" or $t-$ degree and the degree in "∂" or $\partial-$degree. Naturally, those degree notions extend to Q_1. We will note them, respectively, $\deg_t$ and $\deg_\partial$.

2 Primary decomposable subspaces

In order to describe the right ideals of A_1, it is convenient to use the notion of *primary decomposable subspaces* of $\mathbb{k}[t]$.

Recall that $\mathbb{k}$ is not necessarily algebraically closed.

Let $b, h \in R = \mathbb{k}[t]$ and V a k-subspace of R. We set:

$$\mathcal{O}(b) := \{a \in R : a' \in bR\}\ ,$$

where a' denotes the formal derivative of a.

E.g. : one has $\mathcal{O}(t^{n-1}) = \Bbbk + t^n \Bbbk[t]$.

We set also :

$$S(V) := \{a \in R : aV \subseteq V\} \text{ and } \mathcal{C}(R, V) := \{a \in R : aR \subseteq V\} .$$

Clearly $\mathcal{O}(b)$ and $S(V)$ are $\Bbbk$-subalgebras of R. If $b \neq 0$, the Krull dimension of $\mathcal{O}(b)$ is $\dim_{\Bbbk}(\mathcal{O}(b)) = 1$.

The set $\mathcal{C}(R, V)$ is an ideal of R contained in both $S(V)$ and V. Moreover, if $\mathcal{O}(b) \subseteq S(V)$ then $b^2 R \subseteq V$ *i.e.* $b^2 \in \mathcal{C}(R, V)$.

Definition 2 *A non-zero k-vector subspace V of $k[t]$ is said to be primary decomposable (p.d. for short) if $S(V)$ contains a k-subalgebra $\mathcal{O}(b)$, with $b \neq 0$. In this case $\mathcal{C}(R, V)$ is a non zero ideal of R. A p.d. subspace V of $k[t]$ is said irreducible (p.d.i.) if V is not contained in a proper ideal of $k[t]$.*

In [1], R.C. Cannings and M.P. Holland have shown that for p.d. V of R, there is the equality

$$\mathcal{D}(V, V) = \mathrm{End}_{A_1}(\mathcal{D}(R, V)) .$$

It is shown in [2] that for any non zero right ideal I of A_1, there exists $x \in Q_1$ and $\sigma \in \mathrm{Aut}_k(A_1)$ such that :

$$x\sigma(I) = \mathcal{D}(R, \Bbbk[X_n])) ,$$

where $n \in \mathbb{N}$ and $\Bbbk[X_n] := k + t^n k[t]$ is the ring of regular functions on an affine algebraic affine curve X_n.

We will show that the inclusion :

$$H(\mathcal{D}(R, \Bbbk[X_n])) \subseteq H(\mathcal{D}(R, V))$$

where V is a proper p.d.i. subspace of R, implies :

$$\Bbbk[X_n] = V .$$

That result will lead us to the conclusion that the subgroup $H(\mathcal{D}(R, \Bbbk[X_n]))$ is equal to its own normalizer in $\mathrm{Aut}_k(A_1)$.

3 The characteristic elements of a right ideal

The first step in the classification of right ideals of the first Weyl algebra A_1 is the following :

Theorem 3.1 (Stafford [5, lemma 4.2]) *If I is a non-zero right ideal of A_1, then there exist $e, e' \in Q_1$ such that:*

$$(i) \;\; eI \subseteq A_1 \text{ and } eI \cap k[t] \neq \{0\} \; ;$$

$$(ii) \;\; e'I \subseteq A_1 \text{ and } e'I \cap k[\partial] \neq \{0\} \; .$$

With (i) we see that any non-zero right ideal I of A_1 is isomorphic to another ideal I' such that $I' \cap k[t] \neq \{0\}$.

Remark: the element e (*resp.* e') of the theorem is a minimal $\partial-$degree element of I^* (*resp.* a minimal $t-$degree element of I^*).

Corollary 3.2 *There exists an unique element (modulo the multiplicative group $\mathbb{k}^*$) $f \in I$ such that the full set of elements of I with minimum $t-$degree be exactly :*

$$f\mathbb{k}[\partial] \; .$$

In the same way, there exists an unique element (modulo the multiplicative group $\mathbb{k}^*$) $e^* \in I^*$ such that the full set of elements of I^* with minimum $t-$degree be exactly :

$$\mathbb{k}[\partial]e^* \; .$$

Proof : For example, for f : let $e' \in I^*$ such that $e'I \cap \mathbb{k}[\partial] \neq \{0\}$. Let $s \in \mathbb{k}[\partial]$ such that $e'I \cap \mathbb{k}[\partial] = s\mathbb{k}[\partial]$. We can take : $f := e'^{-1}s$. q.e.d.

Definition 3 *The elements $e^* \in I^*$ and $f \in I$ are called the characteristic elements of the ideal I.*

If V is a p.d. subspace and if $I = \mathcal{D}(R, V)$, we will also say that e^* and f are the characteristic elements of the p.d.i. subspace V.

Remark : if e^*, f are the characteristic elements of a right ideal I, then $e^*f \in \mathbb{k}[\partial]$ and :

$$e^*I \cap \mathbb{k}[\partial] = e^*f\mathbb{k}[\partial] \; .$$

E.g. : the characteristic elements of $\Bbbk[X_n] = \Bbbk + t^n\Bbbk[t]$ are :

$$e_n^* := t^{-n}(t\partial) \in \mathcal{D}(\Bbbk[X_n], R) \ ,$$

$$f_n := (t\partial - 1)...(t\partial - (n-1)) \in \mathcal{D}(R, \Bbbk[X_n]$$

and $e_n^* f_n = \partial^n$.

We now recall some important properties of the p.d.i. subspaces V and of the associated right ideals $\mathcal{D}(R, V)$.

Lemma 3.3 *Let I be a right ideal of A_1 such that $I \cap \Bbbk[t] \neq \{0\}$.*

— If $V := I \star 1 := \{d(1) : d \in I\}$, then V is a p.d. subspace of R and

$$I = \mathcal{D}(R, V) \ .$$

— For any p.d. subspace W of R, one has

$$\mathcal{D}(R, W) \star 1 = W \ \textit{and}\ \mathcal{C}(R, W) = \mathcal{D}(R, W) \cap \Bbbk[t] \ .$$

For a proof *cf* [1, theorem §3.2].

Henceforth, θ will denote the $\Bbbk$-automorphism of A_1 such that :

$$\theta(\partial) = t \text{ and } \theta(t) = -\partial \ .$$

According to [1], the above lemma has the following consequence :

Corollary 3.4 *For any non zero right ideal I of A_1,*

i) there is a unique $x \in Q_1$ (modulo the multiplicative group $\Bbbk^*$) and a unique p.d.i. subspace V of R such that $xI = \mathcal{D}(R, V)$;

ii) there is a unique $y \in Q_1$ (modulo the multiplicative group $\Bbbk^*$) and a unique p.d.i. subspace W of R such that : $\theta(yI) = \mathcal{D}(R, W)$.

Now let us give some properties which characterize a p.d.i. subspace V of R.

Proposition 3.5 *Let V be a p.d.i. subspace of R and $m := dim_{\Bbbk} R/V$.*

i) For any $0 \neq d \in \mathcal{D}(R, V)$, $d(R) \subseteq V$ and $\deg_t(d) \geq dim_{\Bbbk} R/V$.

ii) If $0 \neq f \in \mathcal{D}(R, V)$ has minimal t-degree, then $f(R) = V$.

iii) Let e^* and f be the characteristic elements of V. As $e^* \in \mathcal{D}(R,V)^*$ and as $\mathcal{D}(R,V) \cap \Bbbk[t] \neq \{0\}$, we have $e^* \in \Bbbk(t)[\partial]$. More over if :

$$f = t^m c_m(\partial) + t^{m-1} c_{m-1}(\partial) + ... + c_0(\partial)$$

for some $c_i(\partial) \in \Bbbk[\partial]$, and if

$$e^* = b_m(\partial) t^{-m} + u$$

where $b_m(\partial) \in \Bbbk[\partial]$, $u \in \Bbbk(t)[\partial]$ and $deg_t(u) < -m$, then :

$$e^* f = b_m(\partial) c_m(\partial) \ .$$

Those properties have all been proved in [3, remarques 1,2,3].

Remarks :

— Note that the $\Bbbk$-vector space R/V has finite dimension since $\{0\} \neq \mathcal{C}(R,V) \subseteq V$.

— How to calculate f? We take any $f' \in \mathcal{D}(R,V)$ with minimal $t-$degree m, and we expand f' as polynomial in t:

$$f' = t^m a_m(\partial) + t^{m-1} a_{m-1}(\partial) + ... + a_0(\partial)$$

where $a_i(\partial) \in \Bbbk[\partial]$ for all i.

If $p(\partial) := \mathrm{hcf}(a_m(\partial), a_{m-1}(\partial), ..., a_0(\partial))$ then we get $f' = fp(\partial)$ (modulo $\Bbbk^*$).

4 About the automorphisms that stabilizes an ideal

Let I be a right ideal of A_1 such that $I \cap \Bbbk[t] \neq \{0\}$ or $I \cap \Bbbk[\partial] \neq \{0\}$. In this paragraph we wish to determine the automorphisms $\sigma \in Aut_{\Bbbk}(A_1)$ such that $\sigma(I) = I$.

We introduce some particular automorphisms of A_1.

If $p \in R$ we define $\sigma := \exp(\mathrm{ad}(p))$ by :

$$\forall\ d \in A_1,\ \sigma(d) := d + [d,p] + \frac{1}{2!}[[d,p],p] + \frac{1}{3!}[[[d,p],p],p] + ...$$

where $[d,p] := dp - pd$ for all $d \in A_1$.

As the application :

$$A_1 \to A_1\,, \quad d \mapsto [d,p]$$

is a locally nilpotent derivation, σ is a well defined automorphism of the ring A_1. Moreover $\sigma^{-1} = \exp(\mathrm{ad}(-p))$.

The following theorem is fundamental in this paper.

Theorem 4.1 *Let V be a p.d. subspace of R and $\sigma := exp(ad(p))$ where $p \in k[t]$. Then $\sigma(\mathcal{D}(R,V)) = \mathcal{D}(R,V)$ if and only if $p \in S(V)$.*

Proof :

Suppose that $p \in S(V)$.

Let $d \in \mathcal{D}(R,V)$.

Clearly dp and pd are both in $\mathcal{D}(R,V)$, so $[d,p] \in \mathcal{D}(R,V)$, and this implies the first inclusion : $\sigma(D(R,V)) \subseteq \mathcal{D}(R,V)$. In the same way, since $-p \in S(V)$ we get the second inclusion $\sigma^{-1}(D(R,V)) \subseteq \mathcal{D}(R,V)$ and then the equality $\sigma(\mathcal{D}(R,V)) = \mathcal{D}(R,V)$.

Now suppose that $\sigma(\mathcal{D}(R,V)) \subseteq \mathcal{D}(R,V)$.

Let us take an element $f \in \mathcal{D}(R,V)$.

In the formal power series ring $\Bbbk[[T]]$, let

$$\log(1+T) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} T^k\ .$$

In the ring $\Bbbk[[T]]$, there is the equality :

$$\log(1 + (e^T - 1)) = T\ .$$

If we specialize in $T = \mathrm{ad}p$, we get :

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\sigma - \mathrm{Id}_{A_1})^k(f) = [f,p]$$

(the left hand side sum is finite because $f \in A_1$ and $p \in R$).

Now, all the terms $f, \sigma(f), \sigma^2(f), ..., \sigma^n(f), ...$ belong to $\mathcal{D}(R,V)$, so do the terms :

$$(\sigma - \mathrm{Id}_{A_1})^k(f)$$

$(k \geq 1)$. Therefore, $[f,p] \in \mathcal{D}(R,V)$ and we have :

$$[f,p](1) = f(p) - pf(1) \in V$$
$$\Longrightarrow pf(1) \in V \; .$$

But we have :

$$V = \mathcal{D}(R,V) \star 1$$
$$= \{f(1) \,:\, f \in \mathcal{D}(R,V)\}$$

so $pV \subseteq V$ i.e. *$p \in S(V)$.*

q.e.d.

A similar result holds with ∂ instead of t :

Corollary 4.2 *Let I be a right ideal of A_1 such that $I \cap \Bbbk[\partial] \neq \{0\}$. Let W be the p.d. subspace of R such that $\theta(I) = \mathcal{D}(R,W)$. Let $q(\partial) \in \Bbbk[\partial]$ and $\tau := \exp(\mathrm{ad}(q(\partial)))$.*

Then:

$$\tau(I) = I \Longleftrightarrow \theta(q(\partial)) \in S(W) \; .$$

5 The Stafford subgroups

Recall that if I is a right ideal of A_1 the Stafford subgroup of I is noted $H(I)$. In the definition, the notion of isomorphism of right A_1-modules appears. Now, as the ring A_1 is hereditary, if I, J are isomorphic right ideals of A_1, then there exists $x \in Q_1$ such that $xI = J$. So, if I is a right ideal of A_1, then we have :

i) $\forall\, \sigma \in \mathrm{Aut}_{\Bbbk}(A_1),\; H(\sigma(I)) = \sigma H(I)\sigma^{-1}$;

ii) $\forall\, 0 \neq z \in Q_1,\; H(zI) = H(I)$.

We will simply note $H(V) := H(\mathcal{D}(R,V))$ for any p.d. subspace V of R.

Following the above remark, a Stafford subgroup of $\mathrm{Aut}_{\Bbbk}(A_1)$ is of the form $H(V)$ for some p.d.i. subspace of R.

Proposition 5.1 *Let V and W be two p.d.i. subspaces of R.*

If $H(V) \subseteq H(W)$ then :

$$i)\;\; S(V) \subseteq S(W) \text{ and } ii)\;\; \mathcal{C}(R,V) \subseteq \mathcal{C}(R,W) \; .$$

Proof :

i) : Let $p \in S(V)$. *By the theorem 4.1,* $\sigma := \exp(\mathrm{ad}(p)) \in H(V)$, *thus* $\sigma \in H(W)$. *So there exists* $0 \neq a \in Q_1$ *such that* $\sigma(\mathcal{D}(R,W) = a\mathcal{D}(R,W)$. *So* $a\mathcal{D}(R,W) \subseteq A_1$. *In particular,* $a \in \Bbbk(t)[\partial]$. *We have also :*

$$\deg_\partial a \leq \deg_\partial d$$

for all $d \in \mathcal{D}(R,W)$. *Therefore,* $\deg_\partial a = 0$ *and* $a \in \Bbbk(t)$. *But :*

$$\sigma(\mathcal{D}(R,W)) \star 1 = a\mathcal{D}(R,W) \star 1$$

$$= aW$$

thus $aW \subseteq R$ *and* $a \in R$ *because* $RW = R$. *Since* $\sigma(t) = t$, *we have :*

$$a^{-1}\mathcal{D}(R,W) = \sigma^{-1}(\mathcal{D}(R,W))$$

so $a^{-1} \in \Bbbk[t]$ *too. Therefore,* $a \in \Bbbk^*$ *and*

$$\sigma(\mathcal{D}(R,W)) = \mathcal{D}(R,W)$$

$$\Longrightarrow p \in S(W)$$

by the theorem 4.1, again.

ii) : If $a \in \mathcal{C}(R,V)$, *then :*

$$aR \subseteq V$$

$$\Longrightarrow aRV \subseteq V$$

$$\Longrightarrow aR \subseteq S(V)$$

$$\Longrightarrow aR \subseteq S(W)$$

(by i))

$$\Longrightarrow aRW \subseteq W$$

$$\Longrightarrow aR \subseteq W$$

i.e. $a \in \mathcal{C}(R,W)$. *q.e.d.*

Proposition 5.2 *Let I and J be two right ideals of A_1 such that: $\theta(I) = \mathcal{D}(R,V)$ and $\theta(J) = \mathcal{D}(R,W)$ with V and W two p.d.i. subspaces of R. Then :*

$$H(I) \subseteq H(J) \Longrightarrow I \cap k[\partial] \subseteq J \cap \ k[\partial]. .$$

Proof : We apply the proposition 5.1 to $\theta(I)$ and $\theta(J)$. *q.e.d.*

Now, we deduce the following for the caracteristic elements of p.d. subspaces of R :

Corollary 5.3 *Let V and W be two p.d.i. subspaces of R such that $H(V) \subseteq H(W)$.*

If $e_V^* \in \mathcal{D}(R,V)^*$ and $f_V \in \mathcal{D}(R,V)$ are the characteristic elements of V and $e_W^* \in \mathcal{D}(R,W)^*$, $f_W \in \mathcal{D}(R,W)$ are those of W, then :

$$e_V^* f_V \in e_W^* f_W \Bbbk[\partial] \ .$$

Proof :

We have $H(\mathcal{D}(R,V)) = H(e_V^*\mathcal{D}(R,V))$ and $H(\mathcal{D}(R,W)) = H(e_W^*\mathcal{D}(R,W))$, so we have the inclusion:

$$H(e_V^*\mathcal{D}(R,V)) \subseteq H(e_W^*\mathcal{D}(R,W)) \ .$$

Now, we can show that e_V^* is the unique element in Q_1 (modulo $\Bbbk^*$) such that :

$$\theta(e_V^*\mathcal{D}(R,V)) = \mathcal{D}(R,V')$$

for some p.d.i. subspace V' of R.

By the proposition 5.2 above, we have :

$$e_V^*\mathcal{D}(R,V) \cap \Bbbk[\partial] \subseteq e_W^*\mathcal{D}(R,W) \cap \Bbbk[\partial] \ .$$

Since we have $e_V^*\mathcal{D}(R,V) \cap \Bbbk[\partial] = e_V^* f_V \Bbbk[\partial]$ and $e_W^*\mathcal{D}(R,W) \cap \Bbbk[\partial] = e_W^* f_W \Bbbk[\partial]$, we obtain :

$$e_V^* f_V \in e_W^* f_W \Bbbk[\partial] \ .$$

q.e.d.

Now we are ready to prove the main proposition.

We will say that a p.d. subspace of R is *monomial* if it can be generated by monomials.

Proposition 5.4 *Let $V \subset R$ be a proper p.d.i. subspace of R such that $H(\Bbbk[X_n]) \subseteq H(V)$. Then :*

i) V is monomial ;

ii) $\mathcal{C}(R, V) = t^n \Bbbk[t]$;

iii) $V = \Bbbk[X_n]$.

Proof :

If $n = 1$, clearly V would be equal to R, contrary to our hypothesis. So let $n \geq 2$. Let $e^* \in \mathcal{D}(R, V)^*$ and $f \in \mathcal{D}(R, V)$ be the characteristic elements of V.

i) *We have $\Bbbk[X_n] = \Bbbk + t^n \Bbbk[t]$, $\mathcal{C}(R, \Bbbk[X_n]) = t^n \Bbbk[t]$ and $t^n \Bbbk[t] \subseteq \mathcal{C}(R, V)$. In particular, $t^n \in \mathcal{D}(R, V)$ and as $e^* \in \mathcal{D}(R, V)^*$, we have :*

$$e^*,\ f \in \Bbbk[t, t^{-1}, \partial] \ .$$

So, we can use the standard form to describe e^* and f :

$$e^* = t^p a_p(t\partial) + ... + t^q a_q(t\partial)$$

$$f = t^r b_r(t\partial) + ... + t^s b_s(t\partial)$$

for some integers $p \leq q$ and $r \leq s$ and some polynomials $a_i(T)$, $b_j(T) \in \Bbbk[T]$.

The characteristic elements of $\Bbbk[X_n]$ are :

$$e_n^* := t^{-n}(t\partial)\ ,\ f_n := (t\partial - 1)...(t\partial - (n-1))$$

and :

$$e_n^* f_n = \partial^n \ .$$

According to the corollary 5.3, we have :

$$e_n^* f_n \in e^* f \Bbbk[\partial]$$

and so

$$e^* f = \partial^l$$

for some integer $0 \leq l \leq n$. That forces $p = q$ and $r = s$. Therefore $V = f(R)$ is spanned by its monomial terms $t^{p+i} f_p(i)$, $i \geq 0$, and is monomial.

In fact, $1 \leq l \leq n$. Otherwise :

$$e^* f = 1 \Rightarrow e^* \in \Bbbk(t)$$

and :

$$e^*V = e^*\mathcal{D}(R,V)\star 1 \subseteq R$$
$$\Rightarrow e^*R = e^*RV \subseteq R$$
$$\Rightarrow e^* \in \Bbbk[t]$$
$$\Rightarrow e^* = f = 1$$
$$\Rightarrow V = R$$

which is impossible.

*

ii) *As V is monomial and irreducible, $1 \in V$ and so $\Bbbk[X_n] \subseteq V$.*

Suppose $t^{n-1} \in V$ and let us consider the automorphism $\sigma := \exp(\mathrm{ad}(t^{n-1}))$. Clearly t^{n-1} would belong to $S(V)$ since V is monomial and $t^n k[t] \subseteq V$. Then we would have $\sigma \in H(V)$. By applying σ to $H(\Bbbk[X_n])$, we get a new inclusion:

$$\sigma H(\Bbbk[X_n])\sigma^{-1} \subseteq \sigma H(V)\sigma^{-1}$$
$$\iff H(\sigma(\mathcal{D}(R,\Bbbk[X_n]))) \subseteq H(V)\ . \tag{1}$$

But for all $d \in A_1$, for all $j \geq 0$:

$$\underbrace{[...[d,t^{n-1}],t^{n-1}],...,t^{n-1}]}_{j} = dt^{(n-1)j} - jt^{n-1}dt^{(n-1)(j-1)} \bmod t^nA_1\ .$$

So, for all $d \in A_1$, for all $r \in R$:

$$\sigma(d)(r) = \sum_{j\geq 0}\frac{d(t^{(n-1)j}r)}{j!} - \sum_{j\geq 1}\frac{t^{n-1}d(t^{(n-1)(j-1)}r)}{(j-1)!} \bmod t^n\Bbbk[t]$$

(those are in fact finite sums because $d(t^{(n-1)j}r) \in t^n\Bbbk[t]$ for $j >> 0$).

Thus for all $d \in \mathcal{D}(R,\Bbbk[X_n])$, for all $r \in R$, we get :

$$\sigma(d)(r) = (1-t^{n-1})\sum_{j\geq 0}\frac{d(t^{(n-1)j}r)}{j!} \bmod t^n\Bbbk[t]$$
$$\in (1-t^{n-1})\Bbbk[X_n] + t^n\Bbbk[t] = \Bbbk(1-t^{n-1}) + t^n\Bbbk[t]\ .$$

Let $U_n := \Bbbk(1-t^{n-1}) + t^n\Bbbk[t]$. We have just proved :

$$\sigma(\mathcal{D}(R,\Bbbk[X_n])) \subseteq \mathcal{D}(R,U_n)\ .$$

In the same way, we can prove :

$$\sigma^{-1}(\mathcal{D}(R, U_n)) \subseteq \mathcal{D}(R, \Bbbk[X_n]) \ .$$

Therefore, we have exactly :

$$\sigma(\mathcal{D}(R, \Bbbk[X_n])) = \mathcal{D}(R, U_n) \ .$$

Now, the characteristic elements of U_n *are :*

$$e^*_{U_n} := \partial^{n-2} t^{-n}(t\partial) + (-1)^n (n-1)! t^{1-n} \in \mathcal{D}(U_n, R)$$

$$f_{U_n} = (t\partial - 1)...(t\partial - (n-1)) + (-1)^n (n-1)! t^{n-1} \in \mathcal{D}(R, U_n) \ .$$

We have :

$$e^*_{U_n} f_{U_n} = \left(\partial^{n-1} + (-1)^n (n-1)!\right)^2$$

hence :

$$e^*_{U_n} f_{U_n} \notin \partial^l \Bbbk[\partial] = e^* f \Bbbk[\partial] \ .$$

By the corollary 5.3, we deduce :

$$H(\sigma(\mathcal{D}(R, \Bbbk[X_n]))) = H(U_n) \not\subseteq H(V)$$

contrary to (1).

So $t^{n-1} \notin V$ *and we have exactly :*

$$\mathcal{C}(R, V) = t^n \Bbbk[t] \ .$$

$*$

iii) *Now,* $\mathcal{C}(R, V) = t^n k[t]$, *and* $t^{n-1} \notin V$. *For* $n = 2$, *we have already* $V = \Bbbk[X_2]$. *So we will suppose* $n \geq 3$.

Let $1 \leq n_1 < n_2 < ... < n_s < n-1$ *be the integers such that* $t^{n_i} \notin V$. *We will show that* $s = n-2$ *and thus* $V = \Bbbk[X_n]$.

We use again the automorphism $\sigma = \exp(\mathrm{ad}(t^{n-1}))$. *We find :*

$$\sigma(\mathcal{D}(R, V)) = \mathcal{D}(R, V_\sigma)$$

where $V_\sigma := \Bbbk(1 - t^{n-1}) + V \cap t\Bbbk[t]$. *We set :*

$$h(T) := (T - n_1)...(T - n_s)$$

and

$$\lambda := (n-1)!\frac{h(0)}{h(n-1)} .$$

We check that the element

$$g_\sigma := h(t\partial)(t\partial - (n-1))\partial^{n-2} + \lambda t h(t\partial + 1)$$

is an element of $\mathcal{D}(R, V_\sigma)$*. We see that*

$$\begin{aligned} \deg_t(g_\sigma) &= s+1 \\ &= \dim_{\Bbbk} R/V_\sigma \end{aligned}$$

(for example because of the short exact sequence :

$$0 \to V_\sigma/(t\Bbbk[t] \cap V) \to R/(V \cap t\Bbbk[t]) \to R/V_\sigma \to 0 \text{).}$$

So g_σ *has minimum* t*-degree. The element* g_σ *can be expanded as :*

$$g_\sigma = t^{s+1}(\partial^{n-1} + \lambda)\partial^s + t^s b_s(\partial) + ... + t b_1(\partial) - (n-1)h(0)\partial^{n-2}$$

for some polynomials $b_i(T) \in \Bbbk[T]$*.*

Since $\lambda \neq 0$*, the highest common factor of*

$$(\partial^{n-1} + \lambda)\partial^s, b_s(\partial), ..., b_1(\partial), \partial^{n-2}$$

must be some ∂^r *where* $0 \leq r \leq s$*. So if* e^*_σ, f_σ *are the characteristic elements of* V_σ*, we have* $g_\sigma = f_\sigma \partial^r$*. Thus* f_σ *must be equal to :*

$$t^{s+1}(\partial^{n-1} + \lambda)\partial^{s-r} + t^s a_s(\partial) + ... + t a_1(\partial) - (n-1)h(0)\partial^{n-2-r}$$

where $a_i(\partial) := b_i(\partial)\partial^{-r} \in \Bbbk[\partial]$*.*

But :

$$\begin{aligned} & H(\Bbbk[X_n]) \subseteq H(V) \\ \Longrightarrow\ & \sigma H(\Bbbk[X_n])\sigma^{-1} \subseteq \sigma H(V)\sigma^{-1} \\ \Longrightarrow\ & H(U_n) \subseteq H(V_\sigma) \\ \Longrightarrow\ & e^*_{U_n} f_{U_n} \in e^*_\sigma f_\sigma \Bbbk[\partial] \end{aligned}$$

by the corollary 5.3. Now, by the proposition 3.5.iii),

$$e^*_\sigma f_\sigma \in (\partial^{n-1} + \lambda)\partial^{s-r}\Bbbk[\partial] .$$

As a consequence :

$$\left(\partial^{n-1} + (-1)^n (n-1)!\right)^2 \in (\partial^{n-1} + \lambda)\partial^{s-r}\mathbb{k}[\partial]$$

which implies that $s = r$ *and* $\lambda = (-1)^n (n-1)!^2$.

Since $s = r$, *we have :*

$$g_\sigma = f_\sigma \partial^s$$

and :

$$g_\sigma(t^i) = 0$$

for all $0 \leq i \leq s-1$. *Thus* $h(i+1) = 0$ *for* $i = 0, 1, ..., s-1$. *But* $n_1, ..., n_s$ *are the only roots of* h *so :*

$$\forall\, 0 \leq j \leq s,\, n_j = j\ .$$

From the equality $\lambda = (-1)^n (n-1)!$ *we then deduce:*

$$\frac{(-1)^s n_1 ... n_s}{(n-2)...(n-1-s)} = (-1)^n$$

$$\Longleftrightarrow \frac{s!(n-2-s)!}{(n-2)!} = (-1)^{n+s}$$

$$\Longleftrightarrow \binom{n-2}{s} = (-1)^{n+s}$$

$$\Longleftrightarrow s = 0 \text{ and } n \text{ is even or } s = n-2\ .$$

If $s = n-2$, *then* $V = \mathbb{k}[X_n]$ *and the proof is finished. If* $s = 0$ *and* n *is even, then :*

$$V = \mathbb{k} + \mathbb{k}t + ... + \mathbb{k}t^{n-2} + t^n \mathbb{k}[t]$$

with $n \geq 4$.

We set $\sigma := \exp(\mathrm{ad}(t^{n-2}))$. *Then we have :*

$$H(\mathbb{k}[X_n]) \subseteq H(V)$$

$$\Longleftrightarrow \sigma H(\mathbb{k}[X_n])\sigma^{-1} \subseteq \sigma H(V)\sigma^{-1}$$

$$\Longleftrightarrow H(W_n) \subseteq H(V_\sigma)$$

where :

$$W_n = (1 - t^{n-2})\mathbb{k}[X_n] + t^n \mathbb{k}[t]$$

$$= \mathbb{k}(1 - t^{n-2}) + t^n \mathbb{k}[t]$$

and

$$V_\sigma = (1 - t^{n-2})V + t^n \mathbb{k}[t]$$
$$= \mathbb{k} + \mathbb{k}(t - t^{n-1}) + \mathbb{k}t^2 + ... + \mathbb{k}t^{n-2} + t^n \mathbb{k}[t] \ .$$

Now, let $e^*_{W_n}, f_{W_n}$ be the characteristic elements of W_n and e^*_σ, f_σ those of V_σ. Because of the corollary 5.3, we should have :

$$e^*_{W_n} f_{W_n} \in e^*_\sigma f_\sigma \mathbb{k}[\partial] \ . \tag{2}$$

But we can check that :

$$e^*_{W_n} = \left(\frac{\partial^{n-1}}{(n-1)!} + (-1)^{n-1}\partial \right) t^{1-n} + \left(\frac{\partial^{n-2}}{(n-2)!} + (-1)^{n-1} \right) t^{-n} \ ,$$

$$f_{W_n} = \frac{(t\partial - 1)...(t\partial - (n-1))}{(n-1)!} + (-1)^{n-1} t^{n-2}(t\partial - 1) \ ,$$

$$f_\sigma = t \left(\frac{\partial^{n-1}}{(n-1)!} + \partial \right) - \frac{\partial^{n-2}}{(n-2)!} - 1 \ .$$

We deduce that :

$$e^*_{W_n} f_{W_n} = \left(\frac{\partial^{n-1}}{(n-1)!} + (-1)^{n-1}\partial \right)^2$$

$$= \left(\frac{\partial^{n-1}}{(n-1)!} - \partial \right)^2$$

because n is even and :

$$e^*_\sigma f_\sigma \mathbb{k}[\partial] \subseteq \left(\frac{\partial^{n-1}}{(n-1)!} + \partial \right) \mathbb{k}[\partial]$$

which contradicts (2).

Hence $V = \mathbb{k}[X_n]$. *q.e.d.*

Using the description of right ideals of A_1 in [3], we deduce the following :

Corollary 5.5 *For any non principal right ideals I and J , the following equivalences are satisfied:*

$$
\begin{aligned}
& H(I) \subset H(J) \\
\Longleftrightarrow\ & H(I) = H(J) \\
\Longleftrightarrow\ & {}^{\exists}x \in \mathrm{Frac}(A_1),\ {}^{\exists}\sigma \in \mathrm{Aut}_{\Bbbk}(A_1),\ I = x\sigma(J) \ .
\end{aligned}
$$

We now obtain the announced result.

Proposition 5.6 *Let I be any right ideal of A_1. The subgroup $H(I)$ is equal to its own normalizer subgroup in $Aut_{\Bbbk}(A_1)$.*

Proof : By [3], we can suppose $I = \mathcal{D}(R, \Bbbk[X_n])$. Let $\gamma \in Aut(A_1)$ such that

$$\gamma H(\Bbbk[X_n])\gamma^{-1} = H(\Bbbk[X_n]) \ .$$

Then we have

$$H(\gamma(\mathcal{D}(R, \Bbbk[X_n])) = H(\Bbbk[X_n]) \ .$$

We have also : $\gamma(\mathcal{D}(R, \Bbbk[X_n])) \simeq \mathcal{D}(R, V)$ for some p.d.i. subspace V of R.

Thus, we get $H(\Bbbk[X_n]) = H(V)$, so $V = \Bbbk[X_n]$ by the proposition 5.4. Finally, we have $\gamma(\mathcal{D}(R, \Bbbk[X_n])) \simeq \mathcal{D}(R, \Bbbk[X_n])$ and that means $\gamma \in H(I)$.

q.e.d.

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